# Physics 513, Quantum Field Theory

Homework 8

Due Tuesday, 11th November 2003

Jacob Lewis Bourjaily

**Problem 4.1**

We are to consider the problem of the creation of Klein Gordon particles by a classical source. This process can be described by the Hamiltonian

$$H = H_o + \int d^3x \ -j(x)\phi(x),$$

where $H_o$ is the Klein-Gordon Hamiltonian, $\phi(x)$ is the Klein-Gordon filed, and $j(x)$ is a c-number scalar function. Let us define the number $\lambda$ by the relation

$$\lambda = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}|\tilde{j}\,(p)|^2.$$

**a)** We are to show that the probability that the source creates *no* particles is given by

$$P(0) = \left|\langle 0|T\left\{\exp\left[i\int d^4x\ j(x)\phi_I(x)\right]\right\}|0\rangle\right|^2.$$

Without loss of understanding we will denote $\phi \equiv \phi_I$. Almost entirely trivially, we see that

$$H_I = -\int d^3x\ j(x)\phi(x).$$

Therefore,

$$\begin{aligned} P(0) &= \left|\langle 0|T\left\{\exp\left[-i\int dt'\ H_I(t')\right]\right\}|0\rangle\right|^2, \\ &= \left|\langle 0|T\left\{\exp\left[i\int d^4x\ j(x)\phi(x)\right]\right\}|0\rangle\right|^2. \end{aligned}$$



**b)** We are to evaluate the expression for $P(0)$ to the order $j^2$ and show generally that $P(0) = 1-\lambda+\mathcal{O}(\lambda^2)$.

First, let us only consider the amplitude for the process. We can make the naïve expansion

$$\langle 0|T\left\{\exp\left[i\int d^4x\ j(x)\phi(x)\right]\right\}|0\rangle = \langle 0|1|0\rangle + i\int d^4x\ j(x)\langle 0|\phi(x)|0\rangle - \ldots.$$

For every odd power of the expansion, there will be at least one field $\phi_{I_o}$ that cannot be contracted from normal ordering and therefore will kill the entire term. So only even terms will contribute to the expansion. It should be clear that the amplitude will be of the form $\sim 1-\mathcal{O}(j^2)+\mathcal{O}(j^4)-\ldots$. Let us look at the $\mathcal{O}(j^2)$ term. That term is given by

$$\begin{aligned} \langle 0|T\left\{-\tfrac{1}{2}\left(\int d^4x\ j(x)\phi(x)\right)^2\right\}|0\rangle &= -\frac{1}{2}\int d^4x d^4y\ j(x)j(y)\langle 0|T\{\phi(x)\phi(y)\}|0\rangle, \\ &= -\frac{1}{2}\int d^4x d^4y\ j(x)j(y)\ D_F(x-y), \\ &= -\frac{1}{2}\int d^4x d^4y\ \int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip(x-y)}j(x)j(y), \\ &= -\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\ \underbrace{\int d^4x\ j(x)e^{-ipx}}_{\tilde{j}\,(p)}\underbrace{\int d^4y\ j(y)e^{ipy}}_{\tilde{j}\,^*(p)}\frac{i}{p^2-m^2+i\epsilon}, \\ &= -\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}|\tilde{j}\,(p)|^2\frac{i}{p^2-m^2+i\epsilon}, \\ &= -\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\int\frac{dp^0}{(2\pi)}|\tilde{j}\,(p)|^2\frac{i}{p^2-m^2+i\epsilon}. \end{aligned}$$

We know how to evaluate the integral

$$\int \frac{dp^0}{(2\pi)} |\tilde{j}(p)|^2 \frac{i}{p^2 - m^2 + i\epsilon} = \int \frac{dp^0}{(2\pi)} |\tilde{j}(p)|^2 \frac{i}{(p^0)^2 - E_{\mathbf{p}}^2 + i\epsilon},$$
$$= \int \frac{dp^0}{(2\pi)} |\tilde{j}(p)|^2 \frac{i}{(p^0 - E_{\mathbf{p}})(p^0 + E_{\mathbf{p}})}.$$

The function has a simple pole at $p^0 = -E_{\mathbf{p}}$ with the residue

$$\left.\frac{i|\tilde{j}(p)|^2}{p^0 - E_{\mathbf{p}}}\right|_{p^0 = -E_{\mathbf{p}}} = -\frac{i|\tilde{j}(p)|^2}{2E_{\mathbf{p}}}.$$

We know from elementary complex analysis that the contour integral is $2\pi i$ times the residue at the pole. Therefore,

$$-\frac{1}{2}\int \frac{d^3p}{(2\pi)^3} \int \frac{dp^0}{(2\pi)} |\tilde{j}(p)|^2 \frac{i}{p^2 - m^2 + i\epsilon} = -\frac{1}{2}\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} |\tilde{j}(p)|^2,$$
$$= -\frac{1}{2}\lambda.$$

Because we now know the amplitude to the first order of $\lambda$ (or, rather, the second order of $j$), we have shown, as desired, that

$$P(0) = |1 - \tfrac{1}{2}\lambda + \ldots|^2 \sim 1 - \lambda + \mathcal{O}(\lambda^2).$$

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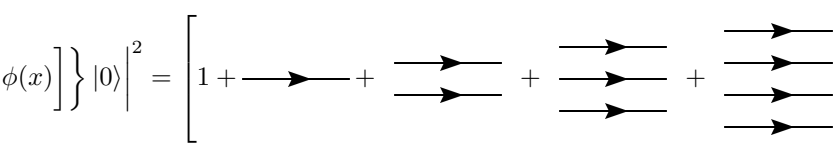
**c)** We must represent the term computed in part (b) as a Feynman diagram and show that the whole perturbation series for $P(0)$ in terms of Feynman diagrams is precisely $P(0) = e^{-\lambda}$.

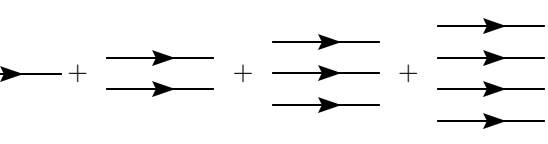
The term computed in part (b) can be represented by  $\equiv -\lambda$. It has two points (neither originated by the source) and a time direction specified (not to be confused with charge or momentum). We can write the entire perturbation series as

$$P(0) = \left|\langle 0|T\left\{\exp\left[i\int d^4x\; j(x)\phi(x)\right]\right\}|0\rangle\right|^2 =$$ 

To get the series we must figure out the correct symmetry factors. If one begins with $2n$ vertices, then $n$ of them must be chosen as 'in'; there are $2^{2n/2} = 2^n$ ways to do this. After that, each one of the 'in' vertices must be paired with one of the 'out' vertices; you can do this $n!$ ways. So the symmetry factor for the term with $n$ uninteracting propagators is

$$S(n) = 2^n \cdot n!.$$

We may now compute the probability explicitly.

$$P(0) =$$ 
$$= \left(\sum_{n=0}^{\infty} \frac{(-\lambda)^n}{2^n n!}\right)^2,$$
$$= \left(\sum_{n=0}^{\infty} \frac{(-\lambda/2)^n}{n!}\right)^2,$$
$$= \left(e^{-\lambda/2}\right)^2,$$

$$\therefore P(0) = e^{-\lambda}.$$

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**d)** Let us now compute the probability that the source creates one particle of momentum $k$. First we should perform this computation to $\mathcal{O}(j)$ and then to all orders using the same trick as in part (c) to sum the series.

Let us calculate the amplitude that a particle is created with the explicit momentum $\mathbf{k}$.

$$\begin{aligned}
&\langle 0|T\left\{\phi_k \exp\left[i\int d^4x\ j(x)\phi(x)\right]\right\}|0\rangle \\
&= i\int d^4x\ j(x)\langle 0|a_{\mathbf{k}}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ipx}+a^\dagger_{\mathbf{p}}e^{ipx}\right)|0\rangle|0\rangle, \\
&= i\int d^4x\ j(x)\langle 0|a_{\mathbf{k}}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}a^\dagger_{\mathbf{p}}e^{ipx}|0\rangle, \\
&= i\int d^4x\ j(x)\langle 0|\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}e^{ipx}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{k})|0\rangle, \\
&= i\int \frac{d^4x}{(2\pi)^4}\frac{j(x)}{\sqrt{2E_{\mathbf{k}}}}e^{ikx}, \\
&= \frac{i\tilde{j}\,(k)}{\sqrt{2E_{\mathbf{k}}}}.
\end{aligned}$$

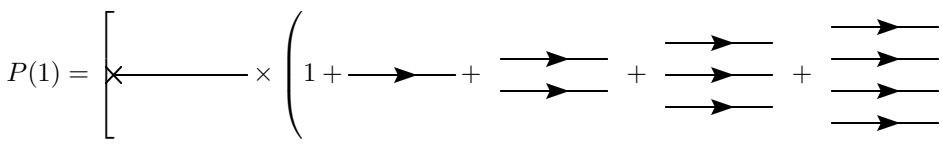
Now, the probability of creating such a particle is the modulus of the amplitude.

$$P(1_{\mathbf{k}}) = \frac{|\tilde{j}(x)|^2}{2E_{\mathbf{k}}}.$$

We can compute the probability that a particle is created with any momentum by simply integrating over all the possible $\mathbf{k}$. This yields

$$P(1) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E_{\mathbf{k}}}|\tilde{j}\,(x)|^2 = \lambda.$$

Therefore in Feynman graphs, $\times\!\!-\!\!-\!\!-\!\!- \equiv i\sqrt{\lambda}$. The entire perturbation in Feynman diagrams is therefore

$P(1) =$  $,$

$$= \left|i\sqrt{\lambda}e^{\lambda/2}\right|^2,$$

$$\therefore P(1) = \lambda e^{-\lambda}.$$

**e)** We are to show that the probability of producing $n$ particles is given by a Poisson distribution.

From part (d) above, we know that each creation vertex on the Feynman diagram must be multiplied by $i\sqrt{\lambda}$. Now, because each of the final products are identical and there are $n!$ ways of arranging them, the symmetry factor in each case is $n!$. The probability is approximated by

$$P(n) \sim \frac{\lambda^n}{n!}.$$

Like we have done before, to get the correct probability, we must take into account the probability that no particle is created. Therefore,

$$P(n) = \frac{\lambda^n e^{-\lambda}}{n!}.$$

**f)** We must show that a poisson distribution given above with parameter $\lambda$ has a norm of 1, an expectation value of $\lambda$, and a variance of $\lambda$.

First, let us compute the norm of the distribution function.

$$\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}e^{\lambda} = e^{-\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!} = e^{-\lambda}e^{\lambda} = 1.$$

The expectation value for the number created is simply,

$$E(n) = \sum_{n=0}^{\infty} \frac{n\lambda^n}{n!} e^{-\lambda} = \lambda e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} = \lambda e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} = \lambda e^{-\lambda} e^{\lambda} = \lambda.$$

To compute the variance, we will use the relation $Var(n) = E(n^2) - E(n)^2$. Let us compute $E(n^2)$.

$$\begin{aligned}
E(n^2) &= \sum_{k=0}^{\infty} n^2 \frac{\lambda^n}{n!} e^{-\lambda}, \\
&= \lambda e^{-\lambda} \sum_{n=1}^{\infty} n \frac{\lambda^{n-1}}{(n-1)!}, \\
&= \lambda e^{-\lambda} \sum_{n=1}^{\infty} ((n-1)+1) \frac{\lambda^{n-1}}{(n-1)!}, \\
&= \lambda e^{-\lambda} \left[ \sum_{n=1}^{\infty} (n-1) \frac{\lambda^{n-1}}{(n-1)!} + \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} \right], \\
&= \lambda e^{-\lambda} e^{\lambda} + \lambda e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-2)!}, \\
&= \lambda^+ \lambda^2 e^{-\lambda} \sum_{n=2}^{\infty} \frac{\lambda^{n-2}}{(n-2)!}, \\
&= \lambda^2 + \lambda.
\end{aligned}$$

Knowing this, it is clear that

$$Var(n) = \lambda^2 + \lambda - \lambda = \lambda.$$

**Problem 4.4**

The cross section for scattering of an electron by the Coulomb field of a nucleus can be computed, to lowest order, without quantizing the electromagnetic field. We will treat the field as a given. classical potential $A_\mu(x)$. The interaction Hamiltonian is then

$$H_I = \int d^3x \; e\bar{\psi}\gamma^\mu \psi A_\mu,$$

where $\psi(x)$ is the usual quantized Dirac field.

**a)** We must show that the $T$-matrix element for an electron scatter to off a localized classical potential is given to the lowest order by

$$\langle p_f | iT | p_i \rangle = -ie\bar{u}(p_f)\gamma^\mu u(p_i) \cdot \tilde{A}_\mu(p_f - p_i).$$

where $\tilde{A}_\mu$ is the Fourier transform of $A_\mu$.

We may compute this contribution directly.

$$\begin{aligned}
\langle p_f | iT | p \rangle &= -i \int d^4x \langle p_f | T\{H_I(x)\} | p_i \rangle, \\
&= -ie \int d^4x \; A_\mu \langle p_f | T\{\bar{\psi}(x)\gamma^\mu \psi(x)\} | p_i \rangle, \\
&= -ie \int d^4x \; A_\mu \langle \overbrace{p_f|\bar{\psi}}(x)\gamma^\mu \overbrace{\psi(x)|p_i}\rangle, \\
&= -ie \int d^4x \; A_\mu(x) \overline{u}^{s'}(p_f)\gamma^\mu u^s(p_i) e^{ix(p_f - p_i)}, \\
&= -ie\overline{u}^{s'}(p_f)\gamma^\mu u^s(p_i) \int d^4x \; A_\mu(x) e^{ix(p_f - p_i)}, \\
&= -ie\overline{u}^{s'}(p_f)\gamma^\mu u^s(p_i) \tilde{A}_\mu(p_f - p_i).
\end{aligned}$$

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**b)** If $A_\mu(x)$ is time independent, its Fourier transform contains a delta function of energy. We therefore define

$$\langle p_f|iT|p_i\rangle \equiv i\mathcal{M}\cdot(2\pi)\delta(E_f-E_i).$$

Given this definition of $\mathcal{M}$, we must show that the cross section for scattering off a time-independent localized potential is given by

$$d\sigma = \frac{1}{v_i}\frac{1}{2E_i}\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}(2\pi)\delta(E_f-E_i)|\mathcal{M}(p_i\to p_f)|^2.$$

From class we know that we can represent an incoming wave packet with momentum $p_i$ in the $z$-direction and impact parameter $b$ by the relation

$$|\psi_b\rangle = \int\frac{d^3p_i}{(2\pi)^3}\frac{1}{\sqrt{2E_{p_i}}}e^{-ibp_i}\psi(p_i)|p_i\rangle.$$

The probability of interaction given an impact parameter is then

$$\begin{aligned}
P(b) &= \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}|\langle p_f|iT|\psi_b\rangle|^2,\\
&= \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\int\frac{d^3p_id^3k}{(2\pi)^6}\frac{1}{\sqrt{2E_{p_i}2E_k}}e^{-ib(p_i-k)}\psi(p_i)\psi^*(k)\langle p_f|iT|p_i\rangle\langle p_f|iT|k\rangle^*,\\
&= \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\int\frac{d^3p_id^3k}{(2\pi)^6}\frac{e^{-ib(p_i-k)}}{\sqrt{2E_{p_i}2E_k}}\psi(p_i)\psi^*(k)(2\pi)^2\delta(E_f-E_{p_i})\delta(E_f-E_k)\mathcal{M}(p_i\to p_f)\mathcal{M}(k\to p_f)^*.
\end{aligned}$$

Therefore,

$$\begin{aligned}
d\sigma &= \int d^2b\ P(b),\\
&= \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\int d^2b\frac{d^3pd^3k}{(2\pi)^6}\frac{e^{-ib(p-k)}}{\sqrt{2E_p2E_k}}\psi(p)\psi^*(k)(2\pi)^2\delta(E_f-E_p)\delta(E_f-E_k)\mathcal{M}(p\to p_f)\mathcal{M}(k\to p_f)^*,\\
&= \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\int\frac{d^3pd^3k}{(2\pi)^6}\frac{\psi(p)\psi^*(k)}{\sqrt{2E_p2E_k}}(2\pi)^2\delta^{(2)}(p_\perp-k_\perp)\delta(E_f-E_p)\delta(E_f-E_k)\mathcal{M}(p\to p_f)\mathcal{M}(k\to p_f)^*,\\
&= \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\frac{1}{|v_i|}(2\pi)\int\frac{d^3pd^3k}{(2\pi)^3}\frac{\psi(p)\psi^*(k)}{\sqrt{2E_p2E_k}}\delta^{(2)}(p_\perp-k_\perp)\delta(p_z-k_z)\delta(E_f-E_p)\mathcal{M}(p\to p_f)\mathcal{M}(k\to p_f)^*,\\
&= \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\frac{1}{|v_i|}(2\pi)\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}|\psi(p)|^2\delta(E_f-E_p)|\mathcal{M}(p\to p_f)|^2,
\end{aligned}$$

With a properly normalized wave function, this reduces directly to (allow me to apologize for the inconsistency with notation. It is hard to keep track of. The incoming momentum $p$ has energy $E_i$.)

$$d\sigma = \frac{1}{v_i}\frac{1}{2E_i}\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}(2\pi)\delta(E_f-E_i)|\mathcal{M}(p_i\to p_f)|^2.$$



Now, let us try to write an expression for $d\sigma/d\Omega$.

$$\begin{aligned}
\int d\sigma &= \int\frac{d^3p_f}{(2\pi)^3}\frac{1}{v_i}\frac{1}{2E_i}\frac{1}{2E_f}(2\pi)\delta(E_f-E_i)|\mathcal{M}|^2,\\
&= \int\frac{p_f^2dp_fd\Omega}{(2\pi)^2}\frac{1}{v_i}\frac{1}{2E_f2E_i}\frac{1}{v_f}\delta(p'-p)|\mathcal{M}|^2,\\
&= \int\frac{d\Omega}{(2\pi)^2}\frac{p^2}{4v_i^2E_i^2}|\mathcal{M}|^2,\\
&= \int d\Omega\frac{1}{16\pi^2}|\mathcal{M}|^2.
\end{aligned}$$

Therefore, we have that

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2}|\mathcal{M}|^2.$$

**c)** We will now specialize to the non-relativistic scattering of a Coulomb potential ($A^0 = Ze/4\pi r$). We must show that in this limit

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 Z^2}{4m^2 v^4 \sin^4(\theta/2)}.$$

Let us first take the Fourier transform of the Coulomb potential.

$$\begin{aligned}\tilde{A}_\mu(\mathbf{k}) &= \frac{Ze}{4\pi}\int d^3r \frac{e^{i\mathbf{kr}}}{\mathbf{r}},\\ &= \frac{Ze}{4\pi}\frac{4\pi}{\mathbf{k}^2},\\ \therefore \tilde{A}_\mu(\mathbf{k}) &= \frac{Ze}{\mathbf{k}^2}.\end{aligned}$$

From part (a) above, we calculated that

$$\begin{aligned}\mathcal{M} &= -ie\overline{u}^{s'}(p_f)\gamma^\mu u^s(p)\tilde{A}_\mu(p_f - p),\\ &= \frac{-ie^2 Z}{(p_f-p)^2}\overline{u}^{s'}(p_f)\gamma^0 u^s(p).\end{aligned}$$

In the nonrelativistic limit, $E >> p$ so we may approximate that

$$\overline{u}^{s'}(p_f)\gamma^0 u^s(p) = u^{s'\dagger}(p_f)u^s(p) = 2E\delta^{s's}.$$

Therefore, our amplitude becomes

$$\mathcal{M} = \frac{-ie^2 Z}{(p_f-p)^2}2E\delta^{s's}.$$

From part (b), we may compute $d\sigma/d\Omega$ directly.

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{4Z^2e^4E62}{16\pi^2(p_f-p)^4},\\ &= \frac{Z^2\alpha^2E^2}{p^4(1-cos\theta)^2},\\ &= \frac{Z^2\alpha^2E^2}{4p^4\sin^4(\theta/2)},\\ &= \frac{Z^2\alpha^2}{4E^2v^4\sin^4(\theta/2)}.\end{aligned}$$

In the nonrelativistic limit, we have that $E^2 \sim m^2$. Therefore we may conclude as desired that

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 Z^2}{4m^2 v^4 \sin^4(\theta/2)}.$$

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